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ANALYTIC SMOOTHING EFFECT FOR THE BENJAMIN-ONO EQUATION

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1. Introduction

We study smoothing effect for the following nonlinear dispersive equation of the Benjamin-Ono type

$$\begin{cases} \partial_t u + \mathcal{H}_x \partial_x^2 u + \partial_x u^2 = 0, & t \in (-T,T), \quad x \in \mathbb{R}, \\ u(0,x) = \phi(x), \end{cases} \tag{1.1}$$

where $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an unknown function and $\mathcal{H}_x$ denotes the Hilbert transform defined by $\mathcal{H}_x v = \mathcal{F} \frac{\xi}{i|\xi|} \hat{v}$ (see [3], [37]). Our problem here is to investigate a sufficient condition of the initial data ϕ on which the solution has regularizing property up to analyticity.

One related problem to (1.1) is the Cauchy problem for the Korteweg - de Vries type

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x u^2 = 0, & t \in (-T,T), \quad x \in \mathbb{R}, \\ u(0,x) = \phi(x). \end{cases} \tag{1.2}$$

This equation appears in the water wave theory and u describes the height of a shallow water wave. There are plenty amount of literatures concerning the study of the KdV equation ([8], [27], [41]). Among others, T. Kato [28] was the first to extract a smoothing effect from the linear part of the KdV equation:

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, & t, x \in \mathbb{R}, \\ v(0,x) = \phi(x). \end{cases} \tag{1.3}$$

He showed the local smoothing effect for the solution to (1.3) as follows:

$$\int_0^T \int_{-R}^R |\partial_x v|^2 dx dt \le C(R,T) \|\phi\|_2^2. \tag{1.4}$$

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[1] This work is a collaboration with Keiichi Kato, and Elena Kaikina, Keiichi Kato and Pavel Naumkin.

This estimate enables us to treat a weak solution of the KdV equation (1.2) in the Sobolev space $H^s(\mathbb{R})$ for $s \geq 3/2$ (See also Sjölin [39], Constantin-Saut [8] and Vega [43] for more general cases). Here $H^s = H^s(\mathbb{R})$ denotes the Sobolev space of order s defined by

$$H^s(\mathbb{R}) = \{f \in \mathcal{S}' : \|f\|_{H^s} < \infty\}, \quad \|f\|_{H^s} \equiv \|\langle\xi\rangle^s \hat{f}\|_2,$$

$\hat{f} = \mathcal{F}f$ is the Fourier transform of f and $\langle\cdot\rangle = (1 + |\cdot|^2)^{1/2}$. Kenig-Ponce-Vega [30] extended the Kato type smoothing effect and moreover discovered a higher order smoothing effect for the inhomogeneous term of the perturbed linear KdV equation. Their estimates to the homogeneous and inhomogeneous part are the following:

$$\|D_x V(t)\phi\|_{L_x^\infty(\mathbb{R};L_T^2)} \leq C\|\phi\|_2, \tag{1.5}$$

$$\|D_x^2 \int_0^t V(t-s)F(s)ds\|_{L_x^\infty(\mathbb{R};L_T^2)} \leq C\|F\|_{L_x^1(\mathbb{R};L_T^2)}, \tag{1.6}$$

where $D_x = \mathcal{H}_x \partial_x$ and $V(t) = e^{-t\partial_x^3}$ denotes the free KdV (Airy) evolution group. Using these estimates (with more extensions), they showed that the KdV equation is well-posed in the Sobolev space $H^{3/4}$. In the series of papers, Bourgain [4] obtained L^2 well-posedness of the KdV equation in the periodic boundary condition. Kenig-Ponce-Vega [31], [32] proved some bilinear estimates involving negative exponent Sobolev spaces and refined the local well-posedness for the Cauchy problem in negative Sobolev spaces $H^s(\mathbb{R})$ for $-3/4 < s$. Those are obtained by the method of the Fourier restriction norm and a sharp estimate for the quadratic nonlinear term in the KdV equation. In fact, the polynomial structure of the nonlinear term has a sort of smoothing effect such as

$$\|\int_0^t V(-s)\partial_x v(s)^2 ds\|_{H_t^b(\mathbb{R};H_x^s)} \leq C\|V(-\cdot)v\|^2_{H_t^b(\mathbb{R};H_x^s)}, \qquad b > \frac{1}{2}. \tag{1.7}$$

On the other hand, a highly regular solution and its smoothing effect were also studied by several authors. T.Kato-Masuda [29] obtained a global smooth solution and analyticity for any point $(t,x) \in \mathbb{R} \times \mathbb{R}$. (Ukai [42] considered a bilinear estimate for the Boltzmann equation in Gevrey and analytic classes.) A smoothing effect up to analyticity or Gevrey class is then proved by Hayashi-K.Kato [16] who obtained an analytic smoothing effect for the nonlinear Schrödinger equation (see also K.Kato-Taniguchi [26]) and de Bouard-Hayashi-K.Kato [10] established analyticity for the KdV equation from Gevrey initial data. Those results are basically obtained by using operators which commute or almost commute to the linear KdV equation.

In [23], we showed that a single point singularity of the initial data at, say the origin, like the Dirac δ measure, immediately disappears after time passes and regularity of the solution to the KdV equation reaches real analyticity in both space and time variables.

To be more precisely, we recall the Fourier restriction space (see [4], [31])

$$\|f\|_{X_b^s} = \left(\iint \langle\tau - \xi^3\rangle^{2b}\langle\xi\rangle^{2s}|\hat{\hat{f}}(\tau,\xi)|^2 d\tau d\xi\right)^{1/2} = \|V(-\cdot)f(\cdot)\|_{H_t^b(\mathbb{R};H_x^s(\mathbb{R}))},$$

where $V(t)$ is the unitary group of the free KdV evolution $e^{-t\partial_x^3}$. The result proved in (K. Kato-Ogawa [23]) for (1.2) was as follows;

Theorem 1.1 ([23]). *Let $-3/4 < s$. Suppose that the initial data $\phi \in H^s(\mathbb{R})$ and satisfy for some $A_0 > 0$,*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!}\|(x\partial_x)^k\phi\|_{H^s} < \infty.$$

Then for some $b \in (1/2, 7/12)$, there exists a unique solution $v \in C((-T,T),H^s) \cap X_b^s$ of the KdV equation (1.2) in a certain time interval $(-T,T)$ and the solution v is time locally well-posed, i.e. the solution continuously depends on the initial data.
Moreover the solution v is analytic at any point $(t,x) \in (-T,0) \cup (0,T) \times \mathbb{R}$.

One can easily see that a typical example of initial data satisfying the assumption of the above theorem is the Dirac delta measure or $p.v.\frac{1}{x}$, where $p.v.$ denotes Cauchy's principal value. Analyticity for the inverse scattering solution with weighted initial data was obtained recently by Tarama [40]. Since our method is based on the fact that the solution is in H^s, we do not know if our result is true globally in time.

Compared with the well-posedness theory to the KdV case, the Benjamin-One equation (1.1) is not yet well-understood. The existence and well-posedness problem of this equation is studied by again T. Kato [27] and also some development was done by Iorio Jr. [21], Ponce [38], Kenig, Ponce and Vega [32] and reference therein. Since the Benjamin-Ono equation has a dispersive structure similar to the Schrödinger equations, we expect that analogous results may hold for the nonlinear problem (1.1).

As a consequence, we observe analytic smoothing effect for the solution to (1.1) with the initial data having a singularity at the origin.

Let $L_s^2(\mathbb{R})$ denote the weighted L^2 defined by

$$\|f\|_{L_s^2} \equiv \|\langle x\rangle^s f\|_2 < \infty.$$

Under the restriction in the weighted Sobolev space, the analytic smoothing effect for the Benjamin-Ono equation can be stated as follows.

Theorem 1.2 ([22]). *Let $s > 3/2$. Suppose that for some $A_0 > 0$, the initial data $\phi \in H^s(\mathbb{R})$ and satisfies*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{((k-1)!)^2}\left(\|(x\partial_x)^k\phi\|_{H^s}^2 + \|(x\partial_x)^k\phi\|_{L_s^2}^2\right) < \infty,$$

(where $0!, (-1)! = 1$*) then there exists a unique solution* $u \in C(\mathbb{R}, H^s)$ *to the nonlinear dispersive equation (1.1) and for any* $(t,x) \in (\mathbb{R}\setminus\{0\}) \times \mathbb{R}$*, there exists some* $A > 0$ *such that we have*

$$|\partial_t^j \partial_x^l u(t,x)| \le C\langle t^{-1}\rangle^{j+l}\langle x\rangle^{2l+3j} A^{j+l}(j+l)!$$

for any $j, l \in \mathbb{N}$*. Namely* $u(t,\cdot)$ *is a real analytic function in both space and time variables* $(t,x) \in \mathbb{R}/\{0\} \times \mathbb{R}$

The existence and uniqueness result of the Benjamin-Ono equation can be found in the articles by Iorio Jr. [21], Ponce [38]. The global well-posedness in time is also discussed by Kenig, Ponce and Vega [32]. Our result is based on those well-posedness results in the Sobolev space $H^s(\mathbb{R})$ with $s > 3/2$. It seems that the well-posedness in a space weaker than $H^{3/2}$ is not well established as far as the author knows. If this is improved such as H^s with $s \le 3/2$, we may extend our result into such a weaker spaces.

We should also emphasize that from the result by Iorio Jr. [21], there is by no means a solution to (1.1) in the weighted space $H^s \cap L^2_s$, where $s > 3$ and there is no solution in $H^s \cap L^2_s$ when $s > 2$ unless the data satisfies $\int \phi(x)dx = 0$. Roughly speaking, this is because the characteristic of the linear part of the Benjamin-Ono equation is not smooth at $\xi = 0$. If we assume that the solution belongs to some weighted L^2 space, then the Fourier transform of the solution has to be smooth (or continuous) around $\xi = 0$. While the evolution $e^{it|\xi|\xi}$ loses the regularity (or integrability) when we take the derivative of evolution by ξ. For example, if we take a derivative more than $5/2$, it loses the L^2 integrability at $\xi = 0$. This contradicts the solution belongs to the weighted space. Since the weight condition on the data is less than $5/2$, the above observation does not really contradict our result. Our solution has a heavy condition on its derivative of solution but not the solution itself. And even more, the solution reaches higher regularity up to space-time analyticity.

Quite recently, a remarkable ill-posedness result was obtained by Molinet, Saut and Tzvetkov [36], where they proved the iteration scheme from the integral equation can not yield the well-posedness in any order of the Sobolev spaces. Since our well-posedness result is based on a quadratic form and multiplication with integration by parts, it is possible to avoid this deficiency.

The essential difference in proving the above results from the case for the nonlinear Schrödinger or KdV equation is due to the appearance of the nonlocal operator $\mathcal{H}_x$.

Since our method is depending on the commutator argument using the generator of the dilation, we reduce the equation into a system of infinitely many equations of the Benjamin-Ono type. It is well known that the Kato type method for the quasi-linear equation ([27]) saves the derivative loss if it is a single equation or having a nice nonlinearity. However the reduced system here is not the case. To fill this gap, we invoke the local smoothing property for the linear dispersive equations. This nature for the dispersive equations was observed by several authors [35], [28], [39], [43], [17] and [8]. Namely, some gauge invariance for the Schrödinger equation also works to save the derivative loss. Hayashi [14] firstly applied the nonlinear gauge transform to obtaining the existence theorem for the nonlinear Schrödinger equation with derivative nonlinearity. Independently the linear case was studied by Doi [11] and his work was developed further by Chihara [6] and Kenig-Ponce-Vega [34]. (see also [18], [15]). Here we apply those local smoothing property under the weight condition to save the derivative loss for the reduced system.

To prove the solution is real analytic in space and time directions, we employ a localization technique. Then it is required to treat the non local term carefully to show the higher regularity. We then introduce a weight function which has an explicit commuting estimate with $\mathcal{H}_x$. This enables us to handle the nonlocal term $\mathcal{H}_x$ in the linear part of the equation. In the following section, we first show the outline of our method and what is the difficulty. Then in the subsequent section we give the local well posedness of the reduced equations and regularity up to the analyticity in space time directions.

2. Method

In this section we give an overview of the whole proof for the Benjamin-Ono case and present some difference from the proof of the former cases in [23] and [24].

Firstly, we introduce the generator of the dilation $P = 2t\partial_t + x\partial_x$ corresponding to the linear part of the dispersive equation. Since the commuting relation with the linear dispersive operator $L \equiv \partial_t + \mathcal{H}_x\partial_x^2$ is

$$[L, P] = 2L,$$

it follows

$$\begin{aligned} &LP^k = (P+2)^k L, \\ &(P+2)^k\partial_x = \partial_x(P+1)^k, \qquad k = 1, 2, \cdots . \end{aligned} \tag{2.1}$$

Applying $P = 2t\partial_x + x\partial_x$ to the equation, we have

$$\begin{cases} \partial_t u + \mathcal{H}_x\partial_x^2 u + \partial_x u^2 = 0, & t, x \in \mathbb{R}, \\ u(0,x) = \phi(x). \end{cases} \tag{2.2}$$

Iteratively, it follows

$$\partial_t(P^k u) + \mathcal{H}_x\partial_x^2(P^k u) = (P+2)^k Lu = -(P+2)^k\partial_x(u^2). \tag{2.3}$$

Then if we set $u_k = P^k u$ and $B_k(u,u) = -(P+2)^k\partial_x u^2$,

$$
\begin{aligned}
(P+1)^l u &= (P+1)^{l-1}Pu + (P+1)^{l-1}u = \cdots \\
&= \sum_{j=0}^{l} \frac{l!}{j!(l-j)!} P^j u,
\end{aligned}
\tag{2.4}
$$

and hence

$$
\begin{aligned}
B_k(u,u) &= -\,(P+2)^k\partial_x(u^2) = -\partial_x(P+1)^k(u^2) \\
&= -\,\partial_x \sum_{l=0}^{k} \binom{k}{l} (P+1)^l u P^{k-l} u \\
&= -\,\partial_x \sum_{k=k_0+k_1+k_2} \frac{k!}{k_0!k_1!k_2!} u_{k_1} u_{k_2}.
\end{aligned}
\tag{2.5}
$$

The important point is that the nonlinear terms $B_k(u,u)$ maintain the bilinear structure similar to the original Benjamin-Ono equation. This is due that the Leibniz rule can be applicable for an operation of P. Thus each of u_k satisfies the following system of equations;

$$
\begin{cases}
\partial_t u_k + \mathcal{H}_x\partial_x^2 u_k = B_k(u,u), & t,x\in\mathbb{R}, \\
v_k(0,x) = (x\partial_x)^k\phi(x).
\end{cases}
\tag{2.6}
$$

Therefore, we consider the following system of dispersive equation and show the well-posedness of the system as well as establishing an estimate for the derivatives

$$
\begin{cases}
\partial_t u_k + \mathcal{H}_x\partial_x^2 u_k = B_k(u,u), & t,x\in\mathbb{R}, \\
u_k(0,x) = \phi_k(x).
\end{cases}
\tag{2.7}
$$

One difficulty to establish the local well-posedness to the above system is that the method of quasi-linear equation by T.Kato [27] does not work well since some of the nonlinear term, say $\partial_x(u_{k_1}u_{k_2})$ does not contains the same function u_k which represents the principal linear part. Therefore it is required to use some kind of smoothing effect from the dispersive property to avoid the derivative loss. Following the argument found in [6], [18] and [15], we consider a variation of the weight function $\mathcal{E}_x(x) = \exp\left(-\int_{-\infty}^{x}\omega^2(y)dy\right)$, where $\omega(x)$ is an appropriately chosen weight function. By commuting $\mathcal{E}_x$ with the linear part, this weight function gains one half derivative under some weight condition, then the local existence and wellposedness for the following infinitely coupled system of Benjamin-Ono type is proved in a proper weighted Sobolev space. We also note that the solution is constructed in a slightly stringent function class such as

$$
\sum_{k=0}^{\infty} \frac{A_1^k}{(k-1)!^2}\|u_k\|^2_{H^s\cap L^2_s} < \infty
$$

than the case of the KdV equation (c.f.[23]), since it is required for the quadratic argument. Then taking $\phi_k = (x\partial_x)^k\phi(x)$, the uniqueness and local well-posedness allow us to say $u_k = P^k u$ for all $k = 0, 1, \cdots$.

Through the process of proving the existence and uniqueness , we obtain the estimate

$$\|P^k u\|_{H^s} \leq CA^k k!.$$

Then we would derive the point-wise derivative estimate by using the equation:

$$\mathcal{H}_x\partial_x^2 P^k u = -\frac{1}{2t}P^{k+1}u + \frac{1}{2t}x\partial_x P^k u + B_k(u,u). \tag{2.8}$$

To treat the second term of the right hand side of (2.8), we employ the localization argument. That is, by $a = a(x)$ we denoted a suitable decaying weight function and can show that

$$\|a\partial_x^l P^k u(t)\|_{H^1(\mathbb{R})} \leq C\langle t^{-1}\rangle^l A^{k+l}(k+l)!, \quad k,l = 0,1,2,\cdots.$$

and then by iterative argument, we can shift from the estimate with the operator P^k to $(t\partial_t)^l$ and conclude

$$\|(t\partial_t)^{l_1}\partial_x^{l_2}u(t)\|_{L^\infty(x_0-\delta,x_0+\delta)} \leq C\langle t^{-1}\rangle^{l_1+l_2}\langle x_0\rangle^{3l_1+2l_2}A^{l_1+l_2}(l_1+l_2)! \tag{2.9}$$

for $l_1, l_2 = 0, 1, 2, \cdots$. In a crucial step for obtaining the above derivative estimates is to treat the nonlocal operator $\mathcal{H}_x$ which is an essential difference from the KdV equation or nonlinear Schrödinger equations. It is well known that the commutator estimate holds between the Hilbert transform and some smooth cut-off function $a(x)$ (c.f., Calderón [5]). However it is now required to show an explicit dependence of the order of the iteration on the constant appeared in the commutator estimate:

$$\|[\mathcal{H}_x, a^k]\partial_x^k\|_{\mathcal{L}(L^2\to L^2)} \leq C_k,$$

where $a^k = a(x)^k$. In order to make it explicit, we choose a particular weight function $a(x) = \langle x\rangle^{-2}$, where $\langle x\rangle = (1+|x|^2)^{1/2}$ and derive an explicit commuting estimate with the Hilbert transform and a^k. By this step, we may use the equation (2.8) to gain the regularity and to show the analyticity (2.9).

3. Construction of the solution

To establish the well-posedness for the system of Benjamin-Ono type equation (2.7), we arrange the equation as follows. By setting $v_k = ((k-1)!)^{-1}u_k$, the equation (2.7) -(2.5) can be reduced as the following slightly simpler system;

$$\begin{cases} \partial_t v_k + \mathcal{H}_x\partial_x^2 v_k = \tilde{B}_k(v,v), & t, x \in \mathbb{R}, \\ u_k(0,x) = \phi_k(x), \end{cases} \tag{3.1}$$

$$\tilde{B}_k(v,v) = -\partial_x \sum_{k=k_0+k_1+k_2} \frac{k}{k_0!k_1 k_2} v_{k_1} v_{k_2}. \tag{3.2}$$

Recall that inside the above summation, k is understood as $k!/(k-1)!$ so that it is 1 if $k=0$.

To recover regularity loss, we introduce a weight function of exponential type. This argument is originally due to Mizohata [35], Kato [28] and Doi [11], later on it is developed by Chihara [6], Hayashi [14] and Kenig-Ponce-Vega [34].

Definition. For $\sigma > 0$ and $b > 0$, we let

$$\begin{aligned} \omega(x) &\equiv b\langle x\rangle^{-1/2-\sigma}, \\ \mathcal{E}_x(x) &\equiv \exp\left(-\int_{-\infty}^{x} \omega^2(y)dy\right). \end{aligned} \tag{3.3}$$

From the definition we see $\|\mathcal{E}_x\psi\| \leq \|\psi\|$ and also the inverse operator $\mathcal{E}_x^{-1}$ is continuous $\|\mathcal{E}_x^{-1}\psi\| \leq e^{Cb}\|\psi\|$.

To see Proposition 4.1 holds, it suffices to show that

Proposition 3.1. *Let $s > 3/2$. Suppose that for some $A_1 > 0$, the initial data $\phi \in H^s(\mathbb{R})$ and satisfies*

$$\sum_{k=0}^{\infty} A_1^k \|\phi_k\|^2_{H^s \cap L^2_s} < \infty,$$

then there exists $T > 0$ such that the system (3.1) with (3.2) is well-posed in

$$\sum_{k=0}^{\infty} A_1^k \left(\|v_k\|^2_{C([0,T];H^s\cap L^2_s)} + \int_0^T \|\omega\mathcal{E}_x\mathcal{H}_x D^{s+\frac{1}{2}} v_k\|_2^2 dt + \int_0^T \|\omega\mathcal{E}_x D^{s+\frac{1}{2}} v_k\|_2^2 dt\right) < \infty.$$

In the next lemma we derive an energy estimate, involving the operator $\mathcal{E}_x$ expressing the smoothing property of the Benjamin-Ono equation originally due to T.Kato [28]. See for some variants in Ponce [38], Doi[11], Chihara [6] and Hayashi [14].

Lemma 3.2. *For a smooth solution u to the linear Benjamin-Ono type equation;*

$$\begin{cases} \partial_t u + \mathcal{H}_x \partial_x^2 u = f, & t, x \in \mathbb{R}, \\ u(0,x) = \phi(x), \end{cases} \tag{3.4}$$

the following inequality holds

$$\frac{d}{dt}\|\mathcal{E}_x u\|_2^2 + 2\left\|\omega\mathcal{E}_x D_x^{1/2} u\right\|_2^2 + 2\left\|\omega\mathcal{E}_x\mathcal{H}_x D_x^{1/2} u\right\|_2^2 \leq 2\left|(\mathcal{E}_x u,\ \mathcal{E}_x f)\right| + C\|u\|_2^2.$$

Proof of Lemma 3.2. Applying operator $\mathcal{E}_x$ to both sides of the linear Benjamin-Ono equation (3.4),

$$\partial_t \mathcal{E}_x u + \mathcal{H}_x \partial_x^2 \mathcal{E}_x u = [\mathcal{H}_x \partial_x^2, \mathcal{E}_x] u + \mathcal{E}_x f.$$

Since $[\partial_x, \mathcal{E}_x] = -\omega^2 \mathcal{E}_x$, we obtain

$$\partial_t \mathcal{E}_x u + \mathcal{H}_x \partial_x^2 \mathcal{E}_x u + 2\omega^2 \mathcal{E}_x D_x u = \left(\omega^4 - \partial_x \omega^2\right) \mathcal{E}_x \mathcal{H}_x u + \partial_x^2 \left[\mathcal{H}_x, \mathcal{E}_x\right] u + \mathcal{E}_x f. \tag{3.5}$$

Therefore multiplying both sides of equation (3.5) by $\mathcal{E}_x u$ and integrating over $\mathbb{R}$, we have

$$\begin{aligned} &\frac{d}{dt} \|\mathcal{E}_x u\|_2^2 + 4\left(\omega^2 \mathcal{E}_x D_x u,\ \mathcal{E}_x u\right) \\ &\quad \leq 2\left|\left((\omega^4 - \partial_x \omega^2)\mathcal{E}_x \mathcal{H}_x u,\ \mathcal{E}_x u\right)\right| + 2\left|\left(\partial_x^2 \left[\mathcal{H}_x,\ \mathcal{E}_x\right] u,\ \mathcal{E}_x u\right)\right| + 2\left|(\mathcal{E}_x f,\ \mathcal{E}_x u)\right|. \end{aligned}$$

For the right hand side, it holds $\|(\omega^4 - \partial_x \omega^2)\|_\infty \leq C$ and we have

$$\begin{aligned} &\left\|\left(\omega^4 - \partial_x \omega^2\right) \mathcal{E}_x \mathcal{H}_x u\right\|_2 \leq C \|u\|_2, \\ &\left\|\partial_x^2 \left[\mathcal{H}_x, \mathcal{E}_x\right] u\right\|_2 \leq C \|u\|_2. \end{aligned}$$

Hence it follows that

$$\frac{d}{dt} \|\mathcal{E}_x u\|_2^2 + 4\left(\omega^2 \mathcal{E}_x D_x u,\ \mathcal{E}_x u\right) \leq 2\left|(\mathcal{E}_x f, \mathcal{E}_x u)\right| + C \|u\|_2^2.$$

On the other hand, we find

$$\begin{aligned} 4\left(\omega^2 \mathcal{E}_x D_x u,\ \mathcal{E}_x u\right) =& 4\left(D_x \omega \mathcal{E}_x u,\ \omega \mathcal{E}_x u\right) - 4\left(\left[D_x, \omega \mathcal{E}_x\right] u,\ \omega \mathcal{E}_x u\right) \\ \geq& 2\left\|D_x^{1/2} \omega \mathcal{E}_x u\right\|_2^2 + 2\left\|\mathcal{H}_x D_x^{1/2} \omega \mathcal{E}_x u\right\|_2^2 - C\|u\|_2^2 \\ \geq& 2\left\|\omega \mathcal{E}_x D_x^{1/2} u\right\|_2^2 + 2\left\|\omega \mathcal{E}_x \mathcal{H}_x D_x^{1/2} u\right\|_2^2 - C\|u\|_2^2. \end{aligned} \tag{3.6}$$

This proves Lemma 3.2. ◻

We give the weighted nonlinear estimate which is the key estimate for proving the well posedness.

Lemma 3.3. *For $0 < \sigma < 1/6$, let $s = \dfrac{3}{2} + 3\sigma$, $\delta = \dfrac{1}{2} + \sigma$ and $\omega(x) = b\langle x\rangle^{-\delta}$ for some $b > 0$. For u, v and $w \in H^s \cap L_s^2$, then we have*

(3.7)

$$\begin{aligned} \left|(\mathcal{E}_x u, \mathcal{E}_x D_x^s \partial_x (vw))\right| \leq& Cb^{-2} \|w\|_{H^s \cap L_s^2} \left(\left\|\omega \mathcal{E}_x D_x^{s+\frac{1}{2}} \mathcal{H}_x v\right\|_2 + \left\|\omega \mathcal{E}_x D_x^{s+\frac{1}{2}} v\right\|_2\right) \left\|\omega \mathcal{E}_x D_x^{\frac{1}{2}} u\right\|_2 \\ &+ Cb^{-2} \|v\|_{H^s \cap L_s^2} \left(\left\|\omega \mathcal{E}_x D_x^{s+\frac{1}{2}} \mathcal{H}_x w\right\|_2 + \left\|\omega \mathcal{E}_x D_x^{s+\frac{1}{2}} w\right\|_2\right) \left\|\omega \mathcal{E}_x D_x^{\frac{1}{2}} u\right\|_2 \\ &+ Cb^{-1} \|v\|_{H^s \cap L_s^2} \|w\|_{H^s \cap L_s^2} \left\|\omega \mathcal{E}_x D_x^{\frac{1}{2}} u\right\|_2 + C\|u\|_2 \|v\|_{H^s \cap L_s^2} \|w\|_{H^s \cap L_s^2}. \end{aligned}$$

For the proof, see [22].

Proof of Proposition 3.1. Let

$$M = 4\left(\sum_{k=0}^{\infty} A_1^k \|\phi_k\|_{H^s \cap L_s^2}^2\right)^{1/2},$$

for $s \in \left(\frac{3}{2}, \frac{5}{3}\right)$ we choose to be such that $s = \frac{3}{2} + 3\sigma$. We define a closed subset of the complete metric space as follows. For $T > 0$ which is determined later, we let

(3.8)

$$\begin{aligned} X_M = \Bigg\{ & f = (f_0, f_1, \cdots); f_k \in C([0,T]; H^s \cap L_s^2), \\ & \|\!|f\|\!|_X \equiv \left(\sum_{k=0}^{\infty} A_1^k \sup_{t\in[0,T]} \|f_k(t)\|_{H^s\cap L_s^2}^2\right) \\ & + \left(\sum_{k=0}^{\infty} A_1^k \int_0^T \left(\|\omega \mathcal{E}_x D_x^{s+\frac{1}{2}} f_k(t)\|_2^2 + \|\omega \mathcal{E}_x \mathcal{H}_x D_x^{s+\frac{1}{2}} f_k(t)\|_2^2\right) dt\right)^{1/2} \le M \Bigg\} \end{aligned}$$

Then we introduce a map Φ on X_M as follows; for any $w \in X_M$, $\Phi(w) = v = (v_0, v_1, \cdots)$, where v solves the linear Benjamin-Ono type equation;

$$\begin{cases} \partial_t v_k + \mathcal{H}_x \partial_x^2 v_k = \tilde{B}_k(w,v), & t, x \in \mathbb{R}, \\ v_k(0,x) = \phi_k(x) \end{cases} \tag{3.9}$$

with

$$\tilde{B}_k(w,v) = -\partial_x \sum_{k=k_0+k_1+k_2} \frac{k}{k_0! k_1 k_2} v_{k_1} w_{k_2}. \tag{3.10}$$

Then we claim that the map Φ is contraction from X_M into X_M. To this end, we apply the operator D_x^s to equation (3.9),

$$\partial_t D_x^s v_k + \mathcal{H}_x \partial_x^2 D_x^s v_k = D_x^s \tilde{B}_k(w,v)$$

and from Lemma 3.2 we have the energy type estimate;

$$\begin{aligned} & \frac{d}{dt} \|\mathcal{E}_x D_x^s v_k\|_2^2 + 2\left\|\omega \mathcal{E}_x D_x^{s+\frac{1}{2}} v_k\right\|_2^2 + 2\left\|\omega \mathcal{E}_x \mathcal{H}_x D_x^{s+\frac{1}{2}} v_k\right\|_2^2 \\ \le & C\left|\left(\mathcal{E}_x D_x^s v_k, \mathcal{E}_x D_x^s \tilde{B}_k(w,v)\right)\right| + C\|\mathcal{E}_x D_x^s v_k\|_2^2. \end{aligned} \tag{3.11}$$

Using Lemma 3.3 and integrating the result with respect to $t \in [0,T]$ and taking a sum over k from 0 to ∞, we find

$$
\begin{aligned}
&\sum_{k=0}^{\infty} A_1^k \sup_{t\in[0,T]} \|\mathcal{E}_x D_x^s v_k\|_2^2 + 2\sum_{k=0}^{\infty} A_1^k \int_0^T \left(\left\|\omega \mathcal{E}_x D_x^{s+\frac12} v_k\right\|_2^2 + \left\|\omega \mathcal{E}_x \mathcal{H}_x D_x^{s+\frac12} v_k\right\|_2^2 \right) dt \\
&\quad \le \frac{M^2}{16} + C_1 e^{A_1} b^{-2} \|w\|_X \left(\int_0^T \sum_{k=0}^{\infty} A_1^k \left\|\omega \mathcal{E}_x D_x^{s+\frac12} v_k\right\|_2^2 dt \right)^{1/2} \\
&\quad \times \left(\int_0^T \sum_{k_1=0}^{\infty} A_1^{k_1} \left(\left\|\omega \mathcal{E}_x D_x^{s+\frac12} v_{k_1}\right\|_2^2 + \left\|\omega \mathcal{E}_x \mathcal{H}_x D_x^{s+\frac12} v_{k_1}\right\|_2^2 dt \right) \right)^{1/2} \\
&\quad + C_1 e^{A_1} b^{-2} \|w\|_X \left(\int_0^T \sum_{k=0}^{\infty} A_1^k \left\|\omega \mathcal{E}_x D_x^{s+\frac12} v_k\right\|_2^2 \sum_{k_1=0}^{\infty} A_1^{k_1} \|v_{k_1}\|_{H^s\cap L_s^2}^2 dt \right)^{1/2} \\
&\quad + C\left(e^{A_1/2}\|w\|_X + 1\right) \int_0^T \sum_{k=0}^{\infty} A_1^k \|v_k\|_{H^s\cap L_s^2}^2 dt.
\end{aligned} \tag{3.12}
$$

We now choose sufficiently large $b > 0$ in (3.12) such that $C_1 e^{A_1} b^{-2} M < \frac{1}{2}$. Then it follows from (3.12) that

$$
\begin{aligned}
&\sum_{k=0}^{\infty} A_1^k \sup_{t\in[0,T]} \|\mathcal{E}_x D_x^s v_k\|_2^2 + \int_0^T \sum_{k=0}^{\infty} A_1^k \left(\left\|\omega \mathcal{E}_x D_x^{s+\frac12} v_k\right\|_2^2 + \left\|\omega \mathcal{E}_x \mathcal{H}_x D_x^{s+\frac12} v_k\right\|_2^2 \right) dt \\
&\quad \le \frac{M^2}{4} + C(M+1) \int_0^T \sum_{k=0}^{\infty} A_1^k \|v_k\|_{H^s\cap L_s^2}^2 dt,
\end{aligned} \tag{3.13}
$$

where the constant $C > 0$.

Similar but much simpler way, we obtain

$$
\begin{aligned}
\sum_{k=0}^{\infty} A_1^k \sup_{t\in[0,T]} \|\langle x\rangle^s v_k\|_2^2 &\le \frac{M}{4} + C \sum_{k=0}^{\infty} \int_0^T A_1^k \|\langle x\rangle^s v_k\|_2 \|v_k\|_{H^s\cap L_s^2} dt \\
&\quad + C\|w\|_X \int_0^T \sum_{k=0}^{\infty} A_1^k \|v_k\|_{H^s\cap L_s^2}^2 dt.
\end{aligned} \tag{3.14}
$$

Choosing sufficiently small time $T > 0$ we obtain from (3.13) and (3.14) the estimate

$$
\begin{aligned}
\|v\|_X &= \sum_{k=0}^{\infty} A_1^k \sup_{t\in[0,T]} \|v_k\|_{H^s\cap L_s^2}^2 + \sum_{k=0}^{\infty} A_1^k \int_0^T \left(\left\|\omega \mathcal{E}_x D_x^{s+\frac12} v_k\right\|_2^2 + \left\|\omega \mathcal{E}_x \mathcal{H}_x D_x^{s+\frac12} v_k\right\|_2^2 \right) dt \\
&\le \frac{M}{2}.
\end{aligned} \tag{3.15}
$$

By virtue of (3.15) we see now that the mapping Φ is from X_M into itself. In the same manner we can prove that for w, $\widetilde{w} \in X_T$,

$$
\|v_k - \tilde{v}_k\|_X < \frac{1}{2}\|w - \widetilde{w}\|_X
$$

if $T > 0$ is sufficiently small, where $v = \Phi(w)$ and $\widetilde{v} = \Phi(\widetilde{w})$. Thus Φ is a contraction mapping in X_T. Therefore there exists a unique solution u of the Cauchy problem (3.9) such that

$$\sum_{k=0}^{\infty} A^k \sup_{t\in[0,T]} \|v_k(t)\|^2_{H^s\cap L^2_s} \leq M^2.$$

This proves Proposition 3.1. □

4. Bootstrap argument

We have constructed a weak solution to the dispersive equation (2.7) satisfying the following extra conormal regularity:

$$\|P^k u\|_{H^s} \leq CA_1^k(k-1)! \quad k = 0, 1, \cdots,$$

under the condition to the initial data ϕ:

$$\|(x\partial_x)^k \phi\|_{H^s} \leq CA_1^k k! \quad k = 0, 1, \cdots.$$

Introducing a smooth weight function; $a(x) = \langle x\rangle^{-2}$, where $\langle x\rangle = (1 + |x|^2)^{1/2}$, we firstly derive

$$\|a^l \partial_x^l P^k u(t)\|_{H^1_x(\mathbb{R})} \leq C\langle t^{-1}\rangle^l A_1^{k+l}(k+l)!, \quad k, l = 0, 1, 2, \cdots. \tag{4.1}$$

To this end, we recall the following lemma originally due to Calderón [5] which plays a key role in the first step of the regularity bootstrap argument.

Lemma 4.1 ([5]). *If $\|a^l\partial_x^l f\|_2 \leq CA^l l! \|f\|_2$ for $0 \leq l \leq N-1$, then there exists a constant $A > 0$ such that we have*

$$\|[\mathcal{H}_x, a^N]\partial_x^N f\|_2 \leq CA^N N! \|f\|_2.$$

Based upon the above Lemma 4.1, we proceed to show the regularity. The first step is the following proposition.

Proposition 4.2. *Let u be a solution constructed in Proposition 3.1 satisfying $\|P^k u(t)\|_{H^s} \leq CA^k k!$ for $k = 0, 1, 2, \cdots$. Then we have*

$$\|\langle x\rangle^{-2l}\partial_x^l P^k u(t)\|_{H^1} \leq C_3\langle t^{-1}\rangle^l A_1^{k+l}(k+l)! \tag{4.2}$$

for all $k, l = 0, 1, 2, \cdots$.

See for the proof, [22]. Rest of the proof goes a similar way as in [23]. By the Sobolev embedding theorem, we have from Proposition 4.2 that for any $x_0 \in \mathbb{R}$ and some $\delta > 0$

$$\|\partial_x^l P^k u(t)\|_{L^\infty(I_{x_0})} \leq C\langle t^{-1}\rangle^l \langle x_0\rangle^{2l} A_1^{k+l}(k+l)! \quad k, l = 0, 1, 2, \cdots,$$

where $I_{x_0} = (x_0 - \delta, x_0 + \delta)$. From this pointwise estimate, we forward the second step and the operator P can be translated into the time derivative via $t\partial_t = \frac{1}{2}(P - x\partial_x)$.

This gives the regularity for the solution.

Proposition 4.3. *For $\delta > 0$, we denote $I_{x_0} = (x_0 - \delta, x_0 + \delta)$. Suppose that there exist positive constants C and A_4 such that*

$$\sup_{t\in[t_0-\delta,t_0+\delta]} \|\partial_x^l P^k u(t)\|_{L^\infty(I_{x_0})} \le C_0 \langle t_0^{-1}\rangle^l \langle x_0\rangle^{2l} A_1^{k+l}(k+l)!, \quad k,l = 0,1,2,\cdots. \tag{4.3}$$

Then we have

$$\sup_{t\in[t_0-\delta,t_0+\delta]} \|\partial_t^j \partial_x^l u(t)\|_{L^\infty(I_{x_0})} \le C \langle t_0^{-1}\rangle^{j+l} \langle x_0\rangle^{2l+3j} A_2^{j+l}(j+l)!, \quad j,l = 0,1,2,\cdots, \tag{4.4}$$

where the constants C and A_2 only depend on C_0, A_1, δ and I_{x_0}.

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